In common English, the terms "sequence" and "series" are often interchanged. For example, a "series of events" generally means the same as "a sequence of events". In mathematics, however, they are different creatures.

1. What is a mathematical series and how is it different from a sequence?

FINITE ARITHMETIC SERIES

The sum of a finite arithmetic series is fairly easy to compute—just plug the right information into a formula and the answer pops out. What is very interesting, however, is finding out how a 9 year old boy figured out that formula.

In 1786, Carl Friedrich Gauss was, at just 9 years old, the youngest student in J.G. Büttner's arithmetic class. The following account was written in 1856 by Wolfgang Sartorius, a colleague of Gauss at the University of Göttingen.

It was a drab, low school-room with a worn, uneven floor....

In this class the pupil who first finished his example in arithmetic was to place his slate in the middle of a large table. On top of this the second placed his slate and so on. The young Gauss had just entered the class when Büttner gave out a problem [the summing of an arithmetic series]. The problem was barely stated before Gauss threw his slate on the table with the words (in the low Braunschweig dialect): "There it lies." While the other pupils continued [counting, multiplying and adding], Büttner, with conscious dignity, walked back and forth, occasionally throwing an ironical, pitying glance toward this the youngest of the pupils. ...

At the end of the hour the slates were turned bottom up. That of the young Gauss with one solitary figure lay on top. When Büttner read out the answer, to the surprise of all present that of young Gauss was found to be correct, whereas many of the others were wrong.

So how did Gauss do it? One theory is that he imagined writing out the numbers 1 to 100 in a single line, and then writing them again in the reverse order. It was then obvious that each pair of values added to 101.

\[
\begin{align*}
1 & \quad + \quad 2 \quad + \quad 3 \quad + \quad \ldots \quad + \quad 98 \quad + \quad 99 \quad + \quad 100 \\
+ & \quad 100 \quad + \quad 99 \quad + \quad 98 \quad + \quad \ldots \quad + \quad 3 \quad + \quad 2 \quad + \quad 1 \\
\hline
101 & \quad + \quad 101 \quad + \quad 101 \quad + \quad \ldots \quad + \quad 101 \quad + \quad 101 \quad + \quad 101
\end{align*}
\]

One-hundred copies of 101 is 10,100. That's double the answer he wanted, so he divided 10,100 by 2 and got 5050.
Another possibility is that he recognized that the integers could be stacked into triangular numbers. Flipping a copy of the triangle upside down would form a rectangle 1 unit wider than it is tall. Half the area of the rectangle would be the desired answer.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

which works for any finite arithmetic series.

2. Find $S_{50}$ if $a_n = 2 + 3(n - 1)$

3. Find $S_{85}$ if $a_n = 900 - 4(n - 1)$

FINITE GEOMETRIC SERIES

The formula for a finite geometric series can be found in a way similar to what Gauss did with finite arithmetic series. The formula is

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$ 

4. Suppose $a_n = 2(1.6)^{n-1}$. Find

(a) $S_{10} = \quad$

(b) $S_{20} = \quad$

(c) $S_{50} = \quad$
5. If \( a_n = 2(0.8)^{n-1} \), find

(a) \( S_{10} = \)

(b) \( S_{20} = \)

(c) \( S_{50} = \)

INFINITE GEOMETRIC SERIES

Notice that in #4 the values kept getting larger as we added more term, but in #5 the values of the sums appeared to level off. The sum will level off (or converge) **only if** \( |r| < 1 \), in which case \( r^n \rightarrow 0 \) as \( n \rightarrow \infty \)

and the formula \( S_n = \frac{a_1(1 - r^n)}{1 - r} \) reduces to \( S_\infty = \frac{a_1}{1 - r} \).

6. Find \( S_\infty \) if \( a_n = 3(0.6)^{n-1} \).

7. Find \( S_\infty \) if \( a_n = 15(-0.8)^{n-1} \).

SIGMA NOTATION

Suppose we want to communicate the idea of the sum of the first 18 terms of the sequence \( a_n = 3 + 2(n-1) \).

One option is to write out all of the terms

\[ 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 + 33 + 35 + 37 \].

The disadvantage to this method is obvious—it takes up too much space. Another way is to say we want

\[ S_{18} \quad \text{if} \quad a_n = 3 + 2(n-1) \].

The disadvantage here is that we have two equations and the connection between \( S_{18} \) and \( a_n \) is not clear. If another sequence was listed, say \( b_n = 2 \cdot 5^{n-1} \), then it would be hard to tell if \( S_{18} \) applied to \( a_n \) or \( b_n \).

Another drawback to using \( S_{18} \) is that it always represents the sum of the **first** 18 terms. If we wanted to start the sum with the 2nd or 3rd terms we’d have a bit of trouble.
Luckily, there is a compact notation that incorporates both the formula for the sequence and the number of terms we want to sum. It’s often called “sigma” notation because we use the Greek letter \( \Sigma \) (sigma) to indicate a sum. Using sigma notation, our example would be written as
\[
\sum_{i=1}^{18} 3 + 2(i-1) .
\]

8. Write \( 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 \) in sigma notation.

9. Write \( S_{25} \) for \( a_n = -3(0.6)^{n-1} \) in sigma notation.

Sigma notation has the added benefit of allowing us to start our sum with any term in the sequence, whereas \( S_n \) always starts with \( a_1 \).

10. Expand \( \sum_{i=1}^{5} 2^i \).

Just because we’ve switched notations, doesn’t mean that the formulas we had earlier don’t work. As long as the sum starts with \( a_1 \) they will still work.

11. Evaluate \( \sum_{k=1}^{\infty} 4(0.5)^{k-1} \).

12. Evaluate \( \sum_{k=1}^{20} 6 - 2(k - 1) \).

13. Suppose you put 1 penny on the first square of a chess board, 2 on the second, 4 on the third, 8 on the fourth, and continued doubling until you got to the 64th square. How much money, in dollars, would be on the chess board? Is this a lot of money? Keep in mind that the national debt is roughly \( 1.2 \times 10^{13} \).

14. After college Bob got a job earning $25,000 with 5% raises each year. 50 years later he retired from that same job. How much money did he earn over his entire career?